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CIRCLE OF SARKISOV LINKS ON A FANO THREEFOLD

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Abstract. It is known that any birational map between Mori fibre spaces factors through a sequence of Sarkisov links [12, 18]. Finding Sarkisov links, or proving their nonexistence, is a first step towards understanding of the group of birational automorphisms of a given Mori fibre space, as these links give information about the generators of this group. By now, there are many examples of Sarkisov links on Mori fibre spaces. The relation between these generators has been recently studied in [26]. In this article we investigate a nontrivial case of such relation, as a circle of Sarkisov links. We describe two different Sarkisov factorizations on a quasi-smooth Fano threefold, to a cubic surface fibration over the projective line; each of these links is a composition of two Sarkisov links. Another Fano threefold is realised at the end of one of the links. We obtain the Sakisov links by playing 2-ray games via variation of geometric invariant theory. In each link we perform a weighted blow up of a quotient singularity, and recover the associate Cox ring and run the 2-ray game. It follows that this Fano variety is neither rational nor birationally rigid and admits at least three different Mori structures.

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1. Introduction

A main goal in algebraic geometry is to have a classification of algebraic varieties up to birational equivalence. While the answer to this problem is a classical result in dimension one, and is more or less settled in dimension two, it is far from being complete in dimension three. A breakthrough in this direction was achieved by the proof of the existence and termination of the Minimal Model Program (MMP), see [28].

. The outcome of MMP, for a given algebraic variety of dimension three, so-called "threefold", is either a minimal model, that is a variety with nef canonical bundle, or is a Mori fibre space. A Mori fibre space is a variety X together with a morphism $\varphi\colon X \to S$ such that

(i) X is $\mathbb{Q}$-factorial and has at worst terminal singularities,
(ii) $-K_X$, the anti-canonical class of X, is φ-ample,
(iii) and X/S has relative Picard number 1.

Mori fibre spaces in dimension three form three classes, decided by the dimension of S:

(1) Fano varieties, when $\dim S = 0$,
(2) del Pezzo fibrations, when $\dim S = 1$,

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(3) and conic bundles, when $\dim S = 2$.

Concerning the Fano case, the unknown case compare to the lower dimensions, there are two natural questions to tackle: construct all possible Fano threefolds (classification), and find the relations between these models. The classical approach to the classification of Fano threefolds can be found in [24]. The mdern approach, however, is to view these objects as varieties embedded in weighted projective spaces, via studying the graded ring $R(X, -K_X)$. There are 95 families of Fano threefolds embedded in a weighted projective space as hypersurfaces, see [21]. Similarly, there are 85 families in codimension two, 70 candidates in codimension three and 145 candidates in codimension four, see [3,4,7–9] for explicit construction and description of the models or [6] for the database. The next step, as we discussed, is to study birational relations between these models.

It is known that Fano varieties in dimension one and two are rational, that is birational to the projective space in that dimension. There are by now many examples of Fano varieties in dimension three that are not rational. In fact, many of these models are "birationally rigid", that is somehow the opposite of being rational.

Definition 1.1. A Mori fibre space $X \to S$ is *birationally rigid* if for any birational map $f\colon X \dashrightarrow X'$ to another Mori fibre space $X' \to S'$, there exists a birational selfmap $\alpha\colon X \dashrightarrow X$ such that the composite $f \circ \alpha\colon X \dashrightarrow X'$ is square.

Definition 1.2. Let $\varphi\colon X \to S$ and $\varphi'\colon X' \to S'$ be Mori fibre spaces such that there is a birational map $f\colon X \dashrightarrow X'$. The map f is said to be *square* if there is a birational map $g\colon S \dashrightarrow S'$, which makes the diagram

$$\begin{tikzcd} X \arrow[r, dashed, "f"] \arrow[d, "\varphi"'] & X' \arrow[d, "\varphi'"] \\ S \arrow[r, dashed, "g"] & S' \end{tikzcd}$$

commute and, in addition, the induced birational map $f_L\colon X_L \to X'_L$ between the generic fibres is biregular. In this situation, we say that the two Mori fibre spaces $X \to S$ and $X' \to S'$ are *birational square*.

Definition 1.3. The *pliability* of a Mori fibre space $X \to S$ is the set

$$\mathcal{P}(X/S) = \{\text{Mfs } Y \to T \mid X \text{ is birational to } Y\}/\sim$$

We sometimes use the term pliability to mean the cardinality of this set, when it is finite and denote it by $\mathrm{Pliab}(X)$. In particular, if X is birationally rigid then $\mathrm{Pliab}(X) = 1$.

It was proved in [22] that a smooth quartic threefold (the simplest case of a Fano hypersurface) is birationally rigid. It was later proved that a general Fano hypersurface satisfies this property [14]. It is conjectured that the same property holds for Fano threefolds in codimension two [13, §6.3]. This conjecture is confirmed for the complete intersection of a cubic and a quadric in $\mathbb{P}^5$, see [23,25]. On the other hand, it was shown in [10] that Fano threefolds in codimension three are not birationally rigid.

A great tool in studying birational properties of Mori fibre spaces is the language of Sarkisov program. Roughly speaking, it says that any birational map between two Mori fibre spaces can be factored into a sequence of Sarkisov links, see [12] for the proof in dimension three and [19] for generalization in higher dimensions. A Sarkisov link is a 2-ray game played on X/S that remains entirely within the Mori category. We refer to [12, §2.2] for definition and explanation of Sarkisov program and 2-ray game.

On the other hand, while Sarkisov program explains the generators of the group of birational automorphisms of a Mori fibre space, [26] gives a theoretical description of the relations between the links.

In this article, we use methods of 2-ray game to describe such relation between models of a Fano threefold in codimension three. We explicitly construct 2-ray games between all models that

we obtain and describe the relations between them. All these 2-ray links remain within the Mori category and therefore are Sarkisov links. In all cases, we construct a suitable toric ambient space, and its Cox ring, and then run the 2-ray game on the ambient variety. We show that in each case the 2-ray game restricts to a game on the threefold, hence at all steps we are dealing with Mori dream spaces [20]. The following is the main result.

Theorem 1.4. *Let $X \subset \mathbb{P}(1,1,1,1,2,2,3)$ be a codimension three Fano threefold. If X is general, in particular quasi-smooth, then it has only two singular points, of type $\frac{1}{2}(1,1,1)$ and $\frac{1}{3}(1,1,2)$, and is birational to a Mori fibre space over $\mathbb{P}^1$, with cubic surface fibres. The birational map factors in two different ways through Sarkisov links, one of which includes two links, through a codimension two Fano threefold $Y_{3,3} \subset \mathbb{P}(1,1,1,1,1,2)$. In particular, the Pliability of X is at least three and there is a circle of Sarkisov links between the models. Moreover, X is not rational.*

We work over the field of complex numbers $\mathbb{C}$, and all varieties are considered projective.

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2. The initial model and its singularities

The variety under consideration is a threefold X embedded in the weighted projective space $\mathbb{P}(1,1,1,1,2,2,3)$; for brevity we denote this weighted projective space by $\mathbb{P}$. Let the coordinates of $\mathbb{P}$ be $x, x_1, x_2, x_3, y, y_1, z$. The threefold X is defined by the vanishing of the Pfaffians of a 5×5 skew-symmetric matrix with upper triangle block given by

$$\begin{pmatrix} y & A_3 & y_1 + C_2 & -x_1 \\ & B_3 & D_2 & x \\ & & z & -y_1 \\ & & & x_3 \end{pmatrix}$$

where A and B are general cubic forms, and C and D are general quadratic forms in variables x, x_1, x_2, x_3, see [3] for general construction. In other words X is the vanishing of

$$(1) \qquad \begin{array}{lll} ① & \mathrm{Pf}_{1234}: & yz = AD - (y_1 + C)B \\ ② & \mathrm{Pf}_{1235}: & yy_1 = -xA - x_1 B \\ ③ & \mathrm{Pf}_{1245}: & yx_3 = x(y_1 + C) + x_1 D \\ ④ & \mathrm{Pf}_{1345}: & x_3 A = -y_1(y_1 + C) + x_1 z \\ ⑤ & \mathrm{Pf}_{2345}: & x_3 B = -y_1 D - xz \end{array}$$

Singular locus of the threefold. Note that A, B, C, D are general so that X is quasi-smooth. In particular, the singular locus of X is $\mathrm{Sing}(X) = \{p_y, p_z\}$, where, for example, p_y is the point $(0{:}0{:}0{:}0{:}1{:}0{:}0) \in X \subset \mathbb{P}$. It is a standard verification, using the equations (1), to see that the germ $p_y \in X$ is isomorphic to the terminal singularity of type $1/2(1,1,1)$. Here we explain how this isomorphism is obtained. This notation will be repeatedly used throughout this article without further explanation.

Define the Zariski open subset $U_y \subset \mathbb{P}$, as the complement of the locus $(y = 0)$. This subset is defined by

$$U_y = \mathrm{Spec}[x, x_1, x_2, x_3, y, y_1, z, \frac{1}{y}]^{\mathbb{C}^*}$$

which is isomorphic to the quotient space

$$\mathrm{Spec}[x, x_1, x_2, x_3, y_1, z]/\mathbb{Z}_2$$

where $\mathbb{Z}_2$ acts, on coordinates, by

$$\epsilon \cdot (x, x_1, x_2, x_3, y_1, z) \mapsto (\epsilon x, \epsilon x_1, \epsilon x_2, \epsilon x_3, y_1, \epsilon z)$$

where ϵ is a 2^{nd} root of unity. In other words, $U_y \cong \mathbb{C}^6/Z_2$; a typical case of a quotient singularity.

Note that the point p_y can be viewed as the origin in this quotient. The singular locus of this quotient is a line passing through the origin. We use the notation $1/2(1,1,1,1,0,1)$ for this point, on the sixfold. There are three tangent variables near this point, z, y_1 and x_3, by ①, ② and ③. Hence the germ $p_y \in X$ is isomorphic to the origin in the quotient space $1/2(1,1,1)$. In particular, p_y is an isolated singularity on the threefold. Similarly, one can check that p_z is of type $1/3(1,1,2)$.

Below, in Sections 3 and 4, we construct two birational maps, starting by Sarkisov links initiated by blowing up these two points.

Definition 2.1. The *index* of a singular point of type $1/n(a,b,c)$ is defined to be the number n.

At the heart of our calculations lies the theorem of Kawamata that asserts what divisorial contractions are centered at quotient singularities.

Theorem 2.2 (Kawamata [27])**.** *Let* $(p\in X) \cong 1/r(a, r-a, 1)$ *be the germ of a threefold terminal quotient singularity. In particular,* a *and* r *are coprime and* $r \geq 2$*. Suppose that* $\varphi\colon (E \subset Y) \to (\Gamma \subset X)$ *is a divisorial contraction such that* Y *is terminal and* $p \in \Gamma$*. Then* $\Gamma = \{p\}$ *and* φ *is the weighted blow up with weights* $(a, r-a, 1)$*.*

We refer to such operation as "Kawamata blow up".

2.1. **The strategy.** The variety X is embedded in a weighted projective space. We aim to find a toric variety with a divisorial contraction to this weighted projective space, such that the restriction of this map to the birational transform of X is a divisorial contraction to a point X and is locally the Kawamata blow up we are after. The toric variety will have rank 2, the rank of its Picard group. Then we run the 2-ray game on it, following [10] and [1]. Next we check whether this 2-ray game restricts to a 2-ray game on the threefold under study, if so we check to see if it is a Sarkisov link. In similar situations, we blow up other Mori fibre space threefolds embedded in weighted projective spaces or rank 2 toric varieties, and follow this instructions. When the rank of the ambient toric variety is 2 we use techniques of [2], in order to realise the rank 3 toric variety after the blow up and the 2-ray game played on it.

3. Blowing up the index two point

The fan of $\mathbb{P}$, as a toric variety, consists of seven 1-dimensional rays $\{\rho_i\}$ in $\mathbb{Z}^6$, forming a complete fan with six (top dimensional) cones $\sigma_i =< \rho_1, \dots, \widehat{\rho_i}, \dots, \rho_7 >$, with a single relation between the rays

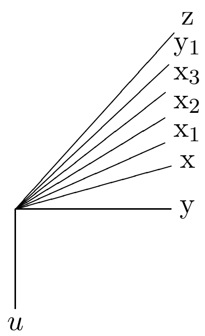
$$\rho_1 + \rho_2 + \rho_3 + \rho_4 + 2\rho_5 + 2\rho_6 + 3\rho_7 = 0,$$

where the coefficients are indicated by the weights that define $\mathbb{P}$. Adding a new ray and performing the consequent subdivision results in a blow up of this variety. We aim to blow up the point p_y, hence the new ray should be in the cone σ_5, see [17, §2.6]. Let the new ray be ρ_0 and the blow up variety denoted by $\mathfrak{X}$. By what we said before, some multiple of ρ_0 can be written as the positive sum of other rays other than ρ_5. Now, we explain how to decide the coefficients in this relation. The Cox ring (set of relations) of $\mathfrak{X}$ is the polynomial ring with eight variables $u, \mathrm{y}, \mathrm{x}, \mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3, \mathrm{y}_1, \mathrm{x}$ associate to the matrix below.

$$\left(\begin{array}{cccccccc} \rho_0 & \rho_5 & \rho_1 & \rho_2 & \rho_3 & \rho_4 & \rho_6 & \rho_7 \\ 0 & 2 & 1 & 1 & 1 & 1 & 2 & 3 \\ -\omega & 0 & \omega_1 & \omega_2 & \omega_3 & \omega_4 & \omega_6 & \omega_7 \end{array}\right)$$

Each column of this matrix indicates a ray in the fan, or represents a variable in the Cox ring. The numerical rows of the matrix represent the relations between the rays, or equivalently the numerical columns represent the bi-degree of the variables in the homogenous coordinate ring (Cox ring) of $\mathfrak{X}$. See [15] for the basic theory and explanation.

Let the new variable, associate to the ray ρ, be u. The GIT chamber of this toric variety has the following shape:



Note that we do not claim $\mathrm{x}, \mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3, \mathrm{y}_1, \mathrm{z}$ are in that order. What is clear is that they all fall in that side of y in comparison with u, because ω and ω_is are all strictly positive. The blow up $\varphi\colon \mathfrak{X} \to \mathbb{P}$ is equivalent to taking the birational map defined by the linear system $|\mathcal{O}(1,0)|$, in other words

$$(u, \mathrm{y}, \mathrm{x}, \mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3, \mathrm{y}_1, \mathrm{z}) \in \mathfrak{X} \mapsto (u^{\frac{\omega_1}{\omega}}\mathrm{x}, u^{\frac{\omega_2}{\omega}}\mathrm{x}_1, u^{\frac{\omega_3}{\omega}}\mathrm{x}_2, u^{\frac{\omega_4}{\omega}}\mathrm{x}_3, \mathrm{y}, u^{\frac{\omega_6}{\omega}}\mathrm{y}_1, u^{\frac{\omega_7}{\omega}}\mathrm{z}) \in \mathbb{P}$$

In order to determine the values of ω and ω_i we use Theorem 2.2. As mentioned before thie map localy near the point p_y on X is sought to be $u^{\frac{1}{2}}\mathrm{x}, u^{\frac{1}{2}}\mathrm{x}_1, u^{\frac{1}{2}}\mathrm{x}_2$. Hence $\omega = 1$ and $\omega_1 = \omega_2 = \omega_3 = 1$. On the other hand, replacing these in the Equations 1 we aim to cancel the highest possible power of u in each equation. This indicates that $\omega_4 = 3, \omega_6 = 4$ and $\omega_7 = 5$. Note that this changes the Cox ring of $\mathfrak{X}$ to

$$\left(\begin{array}{cccccccc} u & \mathrm{y} & \mathrm{x} & \mathrm{x}_1 & \mathrm{x}_2 & \mathrm{z} & \mathrm{y}_1 & \mathrm{x}_3 \\ 0 & 2 & 1 & 1 & 1 & 3 & 2 & 1 \\ -2 & 0 & 1 & 1 & 1 & 5 & 4 & 3 \end{array}\right)$$

However, this matrix defines a stacky fan and not a toric fan (see [2, 5, 16]. However, the suitable toric variety can be obtain by well forming this matrix (see [2]) by subtracting the first row from the second and then dividing by 2, which essentially removes a factor of 2 from the determinant of all 2×2 minors of the matrix above. Hence, the Cox ring of the toric variety $\mathfrak{X}$ is

$$\left(\begin{array}{cccccccc} u & \mathrm{y} & \mathrm{x} & \mathrm{x}_1 & \mathrm{x}_2 & \mathrm{z} & \mathrm{y}_1 & \mathrm{x}_3 \\ 0 & 2 & 1 & 1 & 1 & 3 & 2 & 1 \\ -1 & -1 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right)$$

with irrelevant ideal $I = (u, \mathrm{y}) \cap (\mathrm{x}, \mathrm{x}_1, \mathrm{x}_2, z, \mathrm{y}_1, \mathrm{x}_3)$.

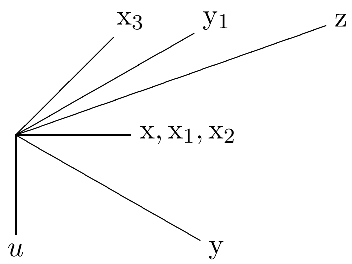
The birational transform of X under this map defines a threefold X_1', that is the Kawamata blow up of X at the point p_y. It is a codimension 3 subvariety of $\mathfrak{X}$ defined by the vanishing of the five equations

$$\begin{array}{lll} (1) & \mathrm{Pf}_{1234}: & \mathrm{yz} = AD - (u\mathrm{y}_1 + C)B \\ (2) & \mathrm{Pf}_{1235}: & \mathrm{yy}_1 = -\mathrm{x}A - \mathrm{x}_1 B \\ (3) & \mathrm{Pf}_{1245}: & \mathrm{yx}_3 = \mathrm{x}(u\mathrm{y}_1 + C) + \mathrm{x}_1 D \\ (4) & \mathrm{Pf}_{1345}: & \mathrm{x}_3 A = -\mathrm{y}_1(u\mathrm{y}_1 + C) + \mathrm{x}_1\mathrm{z} \\ (5) & \mathrm{Pf}_{2345}: & \mathrm{x}_3 B = -\mathrm{y}_1 D - \mathrm{xz} \end{array} \tag{2}$$

where A, B, C, D are the same as before with the following replacements

$$x \mapsto \mathrm{x} \quad , \quad x_1 \mapsto \mathrm{x}_1 \quad , \quad x_2 \mapsto \mathrm{x}_2 \quad , \quad x_3 \mapsto u\mathrm{x}_3$$

Running the 2-ray game on $\mathfrak{X}$ is essentially the variation of Geometric Invariant Theory (vGIT) in the Mori chamber of $\mathfrak{X}$, given by



The 2-ray game follows the diagram

$$\begin{tikzcd}
& \mathfrak{X} \arrow[rr, dashed, "\text{flip}"] \arrow[dl, "\text{blow up}"'] \arrow[dr, "f_0"] & & \mathfrak{X}_1 \arrow[rr, dashed, "\text{flip}"] \arrow[dl, "g_0"'] \arrow[dr, "f_1"] & & \mathfrak{X}_2 \arrow[dl, "g_1"'] \arrow[dr, "\text{blow up}"] & \\
\mathbb{P} & & \mathfrak{T}_0 & & \mathfrak{T}_1 & & \mathbb{P}'
\end{tikzcd}$$

The map f_0 contracts the locus $(\mathrm{z} = \mathrm{y}_1 = \mathrm{x}_3 = 0) \subset \mathfrak{X}$, that is isomorphic to the 3 dimensional scroll

$$\left(\begin{array}{ccccc} u & \mathrm{y} & \mathrm{x} & \mathrm{x}_1 & \mathrm{x}_2 \\ 0 & 2 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 0 \end{array} \right),$$

to the $\mathbb{P}^2_{\mathrm{x}:\mathrm{x}_1:\mathrm{x}_2}$. The map g_0, on the other hand, contracts the 4 dimensional scroll defined by $(u = \mathrm{y} = 0) \subset \mathfrak{X}_1$ to the same $\mathbb{P}^2$. Therefore, the birational map $\mathfrak{X} \dashrightarrow \mathfrak{X}_1$ is a $(-1, -1, 1, 1, 1)$ flip above a $\mathbb{P}^2$. Similarly, the birational map $\mathfrak{X}_1 \dashrightarrow \mathfrak{X}_2$ can be seen as the flip $(3, 5, 1, 1, 1, -1, -2)$, that is the contraction of a $\mathbb{P}(1, 1, 1, 3, 5) \subset \mathfrak{X}_1$ to a point in $\mathfrak{T}_1$ by f_1 and extraction of a $\mathbb{P}(1, 2) \subset \mathfrak{X}_2$ from that point by g_1. The last map is simply the contraction of the divisor $(\mathrm{x}_3 = 0) \subset \mathfrak{X}_2$. It can be written explicitly by the linear system $|\mathcal{O}(2, 1)|$, that is

$$(u, \mathrm{y}, \mathrm{x}, \mathrm{x}_1, \mathrm{x}_2, \mathrm{z}, \mathrm{y}_1, x_3) \in \mathfrak{X}_2 \mapsto (\mathrm{y}_1, \mathrm{x}_3\mathrm{x}, x_3\mathrm{x}_1, \mathrm{x}_3\mathrm{x}_2, x_3^2 u, \mathrm{x}_3\mathrm{z}, x_3^4\mathrm{y}) \in \mathbb{P}(1, 1, 1, 1, 1, 2, 3)$$

Theorem 3.1. *The 2-ray game on $\mathfrak{X}$ restricts to a 2-rag game on X_1'. In particular, X is birational to a complete intersection of two cubics in* $\mathbb{P}(1, 1, 1, 1, 1, 2)$*, and the birational link between them consists of 11 flops followed by a* $(3, 1, -1, -2)$ *flip.*

Proof. Substituting $u = \mathrm{y} = \mathrm{z} = \mathrm{y} - 1 = \mathrm{x}_3 = 0$ in the Equations 2, and then solving in $\mathbb{P}^2_{\mathrm{x}:\mathrm{x}_1:\mathrm{x}_2}$ we obtain 11 points, by Hilbert-Burch theorem. Note that setting $\mathrm{z} = \mathrm{y} - 1 = \mathrm{x}_3 = 0$ gives the same equations, hence f_0 restricted to Y contracts 11 lines to 11 points (on $\mathbb{P}^2$). On the other hand, near any of these points the variable z is a tangent, by 4 and 5. Therefore this variable can be eliminated in a neighborhood of any point in $\mathfrak{X}_1$ that maps to one of those 11 points. Hence g_0 restricted to the threefold (defined by Equations 2) is the contraction of a $\mathbb{P}^1_{\mathrm{y}_1:\mathrm{x}_3}$. Therefore, the first toric flip restricts to 11 flops $(1, 1, -1, -1)$ on X_1', to a threefold X_2'. At the next step, the tangency of variables y, x and x_1 near $p_z \in \mathfrak{T}$ (the image of contractions from f_1 and g_1) allows one to eliminate this variables locally, and hence the toric flip restricts to a threefold flip of type $(3, 1, -1, -2)$, to a threefold X_3'. In particular, the singular point $p_{u\mathrm{z}}$ of type $1/3(1, 1, 2)$ is replaced by the $1/2(1, 1, 1)$ point $p_{\mathrm{x}_3\mathrm{z}}$ after the flip. The last step of the game contracts the divisor $(\mathrm{x}_3 = 0)$ to the point $p_{\mathrm{y}_1} \in \mathbb{P}'$. Restricting this map to Equations 2 shows that the variable y can be globally eliminated on this threefold. After this elimination one can see that the transform of 1 and 2 are in the ideal generated by 4 and 5. Hence the we have a divisorial contraction from X_3' to a threefold Y defined by the vanishing of

$$A + y_1(uy_1 + C) - x_1 z = B + xz + y_1 D = 0$$

in the weighted projective space $\mathbb{P}(1,1,1,1,1,2)$, with variables u, x, x_1, x_2, y_1, z, where A and B are general cubics, and C and D are general quadrics, in variables u, x_1, x_2. In particular Y has two singular points: p_z of quotient type $1/2(1,1,1)$ and p_{y_1}, a cA_1 with local analytic isomorphism

$$(p_{y_1} \in Y) \cong (0 \in (\mathrm{xz} + \mathrm{x}_1^2 + \mathrm{x}_2^2 = 0) \subset \mathbb{C}^4)$$

$\square$

4. Blowing up the index three point

In this section we perform a similar construction for blowing up the point p_z, that has singularity of type $1/3(1,1,2)$. Recall that, X is defined by the vanishing of

$$(3) \qquad \begin{array}{cl} ① & yz = AD - (y_1 + C)B \\ ② & x_1 z = x_3 A + y_1(y_1 + C) \\ ③ & xz = -y_1 D - x_3 B \\ ④ & yy_1 = -xA - x_1 B \\ ⑤ & yx_3 = x(y_1 + C) + x_1 D \end{array}$$

Similar to the previous section, the Cox ring of the sixfold $\mathfrak{X}'$, the toric blow up is

$$\begin{pmatrix} w & \mathrm{z} & \mathrm{x}_2 & \mathrm{x}_3 & \mathrm{y}_1 & \mathrm{y} & \mathrm{x} & \mathrm{x}_1 \\ 0 & 3 & 1 & 1 & 2 & 2 & 1 & 1 \\ -1 & -1 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

The threefold X_1'', the Kawamata blow up of X at the point p_z, is defined by

$$(4) \qquad \begin{array}{cl} ① & \mathrm{yz} = AD - (\mathrm{y}_1 + C)B \\ ② & \mathrm{x}_1\mathrm{z} = \mathrm{x}_3 A + \mathrm{y}_1(\mathrm{y}_1 + C) \\ ③ & \mathrm{xz} = -\mathrm{y}_1 D - \mathrm{x}_3 B \\ ④ & \mathrm{yy}_1 = -\mathrm{x}A - \mathrm{x}_1 B \\ ⑤ & \mathrm{yx}_3 = \mathrm{x}(\mathrm{y}_1 + C) + \mathrm{x}_1 D \end{array}$$

where A, B, C, D are as before with replacements

$$x \mapsto w\mathrm{x} \quad , \quad x_1 \mapsto w\mathrm{x}_1 \quad , \quad x_2 \mapsto \mathrm{x}_2 \quad , \quad x_3 \mapsto \mathrm{x}_3$$

The first step of the ambient 2-ray game restricts to 7 flops, and maps to X_2'', and follows at the next step by a Francia flip $(2,1,-1,-1)$, mapping to $\widetilde{Z}$. At the end, we have a fibration over $\mathbb{P}^1_{\mathrm{x}:\mathrm{x}_1}$ from $\widetilde{Z}$. Using ② and ③ we can eliminate the variable z above all fibres, therefore the new model is the complete intersection of

$$\mathrm{yy}_1 = -\mathrm{x}A - \mathrm{x}_1 B \quad \text{and} \quad \mathrm{yx}_3 = \mathrm{x}(\mathrm{y}_1 + C) + \mathrm{x}_1 D$$

as two hypersurfaces in the toric variety

$$\begin{pmatrix} w & \mathrm{x}_2 & \mathrm{x}_3 & \mathrm{y}_1 & \mathrm{y} & \mathrm{x} & \mathrm{x}_1 \\ 1 & 1 & 1 & 2 & 1 & 0 & 0 \\ -2 & -1 & -1 & -2 & 0 & 1 & 1 \end{pmatrix}$$

Note that this matrix is obtain by removing the variable z and a re-scale using $\mathrm{SL}^*(2,\mathbb{Z})$.

Each fibre is a complete intersection of a quadric with a cubic in $\mathbb{P}(1,1,1,1,2)$, in particular, the generic fibre is a cubic surface. Furthermore, this model has only a singular point of type $1/2(1,1,1)$, at the point $p_{y_1 x_1}$.

5. Mori fibrations

In this section we obtain two other models, both fibred over $\mathbb{P}^1$ with cubic surface fibres, from Y and $\widetilde{Z}$, and show that they are isomorphic. it turns out that these models are square birational to $\widetilde{Z}/\mathbb{P}^1$.

Lemma 5.1. *The blow up of the* $1/2(1,1,1)$ *singular point in* $Y \subset \mathbb{P}(1,1,1,1,1,2)$ *has a Sarkisov link of type* I *to a cubic surface fibration over* $\mathbb{P}^1$.

Proof. Consider the toric variety with Cox ring

$$\begin{pmatrix} v & \mathrm{z} & u & \mathrm{x}_2 & \mathrm{y}_1 & \mathrm{x} & \mathrm{x}_1 \\ 0 & 2 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

With irrelevant ideal $I = (v, \mathrm{z}) \cap (u, \mathrm{x}_2, \mathrm{y}_1, \mathrm{x}, \mathrm{x}_1)$.

Similar to constructions in previous section, it follows that the threefold Y_1' defined by the vanishing of

$$A + y_1(uy_1 + C) - x_1 z = B + xz + y_1 D = 0$$

in this toric variety is the Kawamata blow up of Y at the point p_z, where A, B, C, D are two cubic and two quadrics in $v\mathrm{x}, v\mathrm{x}_1, \mathrm{x}_2, u$. Furthermore, the 2-ray game of the ambient space restricts to a 2-ray game on Y_1', which consists of nine flops followed by a fibration to $\mathbb{P}^1_{\mathrm{x}:\mathrm{x}_1}$. Note that above every point in the base of this fibration the variable z can be eliminated, using the ratio

$$\mathrm{z} = \frac{A + \mathrm{y}_1(u\mathrm{y}_1 + C)}{-\mathrm{x}} = \frac{B + \mathrm{y}_1 D}{\mathrm{x}_1}$$

Hence the new variety, Z, can be viewed as the hypersurface defined by

$$\mathrm{x}_1(A + \mathrm{y}_1(u\mathrm{y}_1 + C)) + \mathrm{x}(B + \mathrm{y}_1 D) = 0$$

in toric variety with Cox ring

$$\begin{pmatrix} v & u & \mathrm{x}_2 & \mathrm{y}_1 & \mathrm{x} & \mathrm{x}_1 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

and irrelevant ideal $I = (\mathrm{x}, \mathrm{x}_1) \cap (v, u, \mathrm{x}_2, \mathrm{y}_1)$. The threefold Z is a fibration of cubic surfaces, over $\mathbb{P}^1_{\mathrm{x}:\mathrm{x}_1}$ with a singular point $p_{\mathrm{y}_1\mathrm{x}}$, which is a cA_1 singularity. □

Lemma 5.2. *The blow up of the* $1/2(1,1,1)$ *singular point in* $\widetilde{Z}$ *has a Sarkisov link of type* II *to another cubic surface fibration over* $\mathbb{P}^1$. *Furthermore, these two models are square birational.*

Proof. Let us recall that $\widetilde{Z}$ is the complete intersection of

$$\mathrm{y}\mathrm{y}_1 = -\mathrm{x}A - \mathrm{x}_1 B \quad \text{and} \quad \mathrm{y}\mathrm{x}_3 = \mathrm{x}(\mathrm{y}_1 + C) + \mathrm{x}_1 D$$

in the toric variety

$$\begin{pmatrix} w & \mathrm{x}_2 & \mathrm{x}_3 & \mathrm{y}_1 & \mathrm{y} & \mathrm{x} & \mathrm{x}_1 \\ 1 & 1 & 1 & 2 & 1 & 0 & 0 \\ -2 & -1 & -1 & -2 & 0 & 1 & 1 \end{pmatrix}$$

with irrelevant ideal $I = (\mathrm{x}, \mathrm{x}_1) \cap (w, \mathrm{x}_2, \mathrm{x}_3, \mathrm{y}_1, \mathrm{y})$. It has a singular point of type $1/2(1,1,1)$ at $p_{\mathrm{x}_1\mathrm{y}_1}$. We aim to blow up this point. Using the techniques introduced in [2], we consider the toric variety of rank three with Cox ring

$$\begin{pmatrix} w & \mathrm{x}_2 & \mathrm{x}_3 & \mathrm{y}_1 & \mathrm{y} & \mathrm{x} & \mathrm{x}_1 & \mathrm{x}' \\ 1 & 1 & 1 & 2 & 1 & 0 & 0 & 0 \\ -2 & -1 & -1 & -2 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 3 & 2 & 0 & -2 \end{pmatrix}$$

with irrelevant ideal $J = (\mathrm{x}, \mathrm{x}_1) \cap (w, \mathrm{x}_2, \mathrm{x}_3, \mathrm{y}_1, \mathrm{y}) \cap (w, \mathrm{x}_2, \mathrm{x}_3, \mathrm{y}, \mathrm{x}) \cap (\mathrm{x}', \mathrm{x}_1) \cap (\mathrm{x}', \mathrm{y}_1)$. Considering the GIT (Mori) chamber of this variety in $\mathbb{Z}^3$, the morphism that corresponds to the wall $\langle e_1, e_2\rangle$ is

$$(w, \mathrm{x}_2, \mathrm{x}_3, \mathrm{y}_1, \mathrm{y}, \mathrm{x}, \mathrm{x}_1, \mathrm{x}') \mapsto (w\mathrm{x}'^{\frac{1}{2}}, \mathrm{x}_2\mathrm{x}'^{\frac{1}{2}}, \mathrm{x}_3\mathrm{x}'^{\frac{1}{2}}, \mathrm{y}_1, \mathrm{y}\mathrm{x}'^{\frac{3}{2}}, \mathrm{x}\mathrm{x}', \mathrm{x}_1)$$

which is, when restricted to $\widetilde{Z}$, the Kawamata blow up of the singular point. However, note that the well-formed model of the toric ambient space is

$$\begin{pmatrix} w & \mathrm{x}_2 & \mathrm{x}_3 & \mathrm{y}_1 & \mathrm{y} & \mathrm{x} & \mathrm{x}_1 & \mathrm{x}' \\ 1 & 1 & 1 & 2 & 1 & 0 & 0 & 0 \\ -2 & -1 & -1 & -2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 & 0 & -1 \end{pmatrix}$$

We aim to play the 2-ray game on this toric variety over the field of rational functions of $\mathbb{P}^1_{\mathrm{x}:\mathrm{x}_1}$, that is the base of the Mori fibration. This can be realised by fixing the action of the $\mathbb{C}^*$ of the second row of this matrix, using a fixed non-zero value of x_1, which leads to the rank two matrix

$$\begin{pmatrix} \mathrm{x} & \mathrm{y} & w & \mathrm{x}_2 & \mathrm{x}_3 & \mathrm{y}_1 & \mathrm{x}' \\ 0 & 1 & 1 & 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 0 & 0 & -1 & -1 \end{pmatrix}$$

and the ideal transforms to $J' = (w, \mathrm{x}_2, \mathrm{x}_3, \mathrm{y}, \mathrm{x}) \cap (\mathrm{x}', \mathrm{y}_1)$. The two ray game of this toric variety, over the base field $\mathbb{C}\{t\}$, restricts to six flops, corresponding to the six points solutions of the general quadric D and general cubic B in $\mathbb{P}^2_{w:\mathrm{x}_2:\mathrm{x}_3}$. The next step is a divisorial contraction. It is more useful to see this contraction on the global model, i.e. the rank three variety. Note that after flops the toric variety has the same Cox ring with irrelevant ideal $J = (\mathrm{x}, \mathrm{x}_1) \cap (w, \mathrm{x}_2, \mathrm{x}_3, \mathrm{y}_1, \mathrm{y}) \cap (\mathrm{y}, \mathrm{x}) \cap (\mathrm{x}', \mathrm{x}_1) \cap (\mathrm{x}', \mathrm{y}_1, w, \mathrm{x}_2, \mathrm{x}_3)$. Rewriting the matrix, using some $\mathrm{SL}^*(3, \mathbb{Z})$ action, in the form

$$\begin{pmatrix} w & \mathrm{x}_2 & \mathrm{x}_3 & \mathrm{y}_1 & \mathrm{y} & \mathrm{x} & \mathrm{x}_1 & \mathrm{x}' \\ 1 & 1 & 1 & 2 & 1 & 0 & 0 & 0 \\ -2 & -1 & -1 & -1 & -1 & 0 & 1 & 1 \\ -1 & -1 & -1 & -3 & 0 & 1 & 0 & -1 \end{pmatrix}$$

Note that the map corresponding to the wall $\langle e_1, e_2\rangle$ in the GIT cone is a morphism that contracts the divisor $(\mathrm{x} = 0)$ to the point $p_{\mathrm{x}'\mathrm{y}}$ in the toric variety with Cox ring

$$\begin{pmatrix} w & \mathrm{x}_2 & \mathrm{x}_3 & \mathrm{y}_1 & \mathrm{y} & \mathrm{x}_1 & \mathrm{x}' \\ 1 & 1 & 1 & 2 & 1 & 0 & 0 \\ -2 & -1 & -1 & -1 & -1 & 1 & 1 \end{pmatrix}$$

with irrelevant ideal $I' = (\mathrm{x}', \mathrm{x}_1) \cap (w, \mathrm{x}_2, \mathrm{x}_3, \mathrm{y}_1, \mathrm{y})$. The blow up (contraction) map is

$$(w, \mathrm{x}_2, \mathrm{x}_3, \mathrm{y}_1, \mathrm{y}, \mathrm{x}, \mathrm{x}_1, \mathrm{x}') \mapsto (w\mathrm{x}, \mathrm{x}_2\mathrm{x}, \mathrm{x}_3\mathrm{x}, \mathrm{y}_1\mathrm{x}^3, \mathrm{y}, \mathrm{x}_1, \mathrm{x}'\mathrm{x})$$

Carrying all these maps on the equation of $\widetilde{Z}$ we end up with a new threefold $\overline{Z}$, defined as the complete intersection of

$$\mathrm{y}\mathrm{y}_1 = -\mathrm{x}'A - \mathrm{x}_1B \quad \text{and} \quad \mathrm{y}\mathrm{x}_3\mathrm{x}' = \mathrm{y}_1 + \mathrm{x}'C + \mathrm{x}_1D$$

in the latter rank two toric variety. Clearly the variable y_1 can be eliminated. Hence, $\overline{Z}$ is the hypersurface

$$\mathrm{y}^2\mathrm{x}_3\mathrm{x}' - \mathrm{x}'\mathrm{y}C - \mathrm{x}_1\mathrm{y}D = -\mathrm{x}'A - \mathrm{x}_1B$$

in the toric variety with Cox ring

$$\begin{pmatrix} w & \mathrm{x}_2 & \mathrm{x}_3 & \mathrm{y} & \mathrm{x}_1 & \mathrm{x}' \\ 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

and irrelevant ideal $\overline{I} = (\mathrm{x}', \mathrm{x}_1) \cap (w, \mathrm{x}_2, \mathrm{x}_3, \mathrm{y})$. It is easy to see that $\overline{Z}$ is a cubic surface fibration over $\mathbb{P}^1_{\mathrm{x}':\mathrm{x}_1}$, and has a cA_1 singularity at the point $p_{\mathrm{y}\mathrm{x}_1}$. □

Proposition 5.3. *The two models obtained in Lemma 5.1 and Lemma 5.2 are isomorphic.*

Proof. This is clear from Lemma 5.1 and Lemma 5.2. □

6. Pliability of the models and the circle

It follows from Theorem 3.1, Lemma 5.1, Lemma 5.2 and Corollary 5.3 that the Fano threefold $X \subset \mathbb{P}(1,1,1,1,2,2,3)$ is birtional to a Fano complete intersection $Y_{3,3} \subset \mathbb{P}(1,1,1,1,1,2)$ and a fibration of cubic surfaces over $\mathbb{P}^1$, up to square birational equivalence. Hence the pliability of this threefold is at least three. The following lemma shows that X is not rational.

Lemma 6.1. *The threefold X and consequently all other models birational to it are non-rational.*

Proof. Considering the models of del Pezzo fibration birational to X, it follows from [11] that these varieties are not rational. □

The diagram below visualizes the relation between all models discussed in this article.

$$
\begin{tikzcd}
 & X_1' \arrow[dl, "\text{Bl up } \frac{1}{2}(1,1,1)"'] \arrow[r, dashed, "11 \text{ flops}", "(1,1,-1,-1)"'] & X_2' \arrow[r, dashed, "\text{flip}", "(3,1,-1,-2)"'] & X_3' \arrow[dr, "\text{Bl up } cA_1"'] & & Y_1' \arrow[dl, "\text{Bl up } \frac{1}{2}(1,1,1)"'] \arrow[r, dashed, "9 \text{ flops}", "(1,1,-1,-1)"'] & Z \arrow[dl, "dP_3"] \arrow[dd, "\cong"] \\
X & & & & Y & \mathbb{P}^1 \arrow[d, phantom, "\square"] & \\
 & X_1'' \arrow[ul, "\text{Bl up } \frac{1}{3}(1,1,2)"] \arrow[r, dashed, "7 \text{ flops}", "(1,1,-1,-1)"'] & X_2'' \arrow[r, dashed, "\text{flip}", "(2,1,-1,-1)"'] & \widetilde{Z} \arrow[urr, "dP_3"] \arrow[rrr, dashed, "\text{Square Birational}"] & & & \overline{Z} \arrow[ul, "dP_3"'] \\
 & & & & Z_1' \arrow[ul, "\text{Bl up } \frac{1}{2}(1,1,1)"] \arrow[r, dashed, "6 \text{ flops}", "(1,1,-1,-1)"'] & Z_2' \arrow[ur, "\text{Bl up } cA_1"] &
\end{tikzcd}
$$

The following question seems natural. It relates the number of singular points on a general Fano threefold in codimension 3 to its pliability.

Question 6.2. Let X be a Fano threefold embedded in a weighted projective space in codimension 3. Let n be the number of different analytic types of singularities that appear in X. Is it true that if X is general then $\mathrm{Pliab}(X) = n+1$?

References

1. Hamid Ahmadinezhad, *On del Pezzo fibrations that are not birationally rigid*, J. Lond. Math. Soc. (2) **86** (2012), no. 1, 36–62. MR 2959294
2. ———, *On pliability of del Pezzo fibrations and Cox rings*, arXiv:1304.4357 (2013), 15 pages.
3. Selma Altınok, *Constructing new K3 surfaces*, Turkish J. Math. **29** (2005), no. 2, 175–192. MR 2142293 (2006d:14037)
4. Selma Altinok, Gavin Brown, and Miles Reid, *Fano 3-folds, k3 surfaces and graded rings*, Topology and geometry: commemorating SISTAG, Contemp. Math., vol. 314, Amer. Math. Soc., Providence, RI, 2002, pp. 25–53. MR MR1941620
5. Lev A. Borisov, Linda Chen, and Gregory G. Smith, *The orbifold Chow ring of toric Deligne-Mumford stacks*, J. Amer. Math. Soc. **18** (2005), no. 1, 193–215 (electronic). MR 2114820 (2006a:14091)
6. Gavin Brown, *Graded ring database*, http://grdb.lboro.ac.uk.
7. Gavin Brown, Michael Kerber, and Miles Reid, *Fano 3-folds in codimension 4, Tom and Jerry. Part I*, Compos. Math. **148** (2012), no. 4, 1171–1194. MR 2956040
8. Gavin Brown and Kaori Suzuki, *Computing certain Fano 3-folds*, Japan J. Indust. Appl. Math. **24** (2007), no. 3, 241–250. MR MR2374989

9. ______, *Fano 3-folds with divisible anticanonical class*, Manuscripta Math. **123** (2007), no. 1, 37–51. MR MR2300058
10. Gavin Brown and Francesco Zucconi, *Graded rings of rank 2 Sarkisov links*, Nagoya Math. J. **197** (2010), 1–44. MR 2649280 (2011f:14020)
11. Ivan Cheltsov, *Nonrational del Pezzo fibrations*, Adv. Geom. **8** (2008), no. 3, 441–450. MR 2427469 (2009m:14016)
12. Alessio Corti, *Factoring birational maps of threefolds after Sarkisov*, J. Algebraic Geom. **4** (1995), no. 2, 223–254.
13. ______, *Singularities of linear systems and 3-fold birational geometry*, Explicit Birational Geometry of 3-folds, vol. 281, London Math. Soc. Lecture Notes Series, Cambridge Univ. Press, Cambridge, 2000, pp. 259–312.
14. Alessio Corti, Aleksandr Pukhlikov, and Miles Reid, *Fano* 3*-fold hypersurfaces*, Explicit birational geometry of 3-folds, London Math. Soc. Lecture Note Ser., vol. 281, Cambridge Univ. Press, Cambridge, 2000, pp. 175–258. MR 1798983 (2001k:14034)
15. David A. Cox, *The homogeneous coordinate ring of a toric variety*, J. Algebraic Geom. **4** (1995), no. 1, 17–50. MR 1299003 (95i:14046)
16. Barbara Fantechi, Etienne Mann, and Fabio Nironi, *Smooth toric Deligne-Mumford stacks*, J. Reine Angew. Math. **648** (2010), 201–244. MR 2774310 (2012b:14097)
17. William Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies, vol. 131, Princeton University Press, Princeton, NJ, 1993, The William H. Roever Lectures in Geometry. MR 1234037 (94g:14028)
18. C. D. Hacon and J. McKernan, *The Sarkisov program*, arXiv:0905.0946 (2009).
19. Christopher D. Hacon and James McKernan, *The Sarkisov program*, arXiv:0905.0946v2.
20. Yi Hu and Sean Keel, *Mori dream spaces and GIT*, Michigan Math. J. **48** (2000), 331–348, Dedicated to William Fulton on the occasion of his 60th birthday. MR 1786494 (2001i:14059)
21. Anthony R. Iano-Fletcher, *Working with weighted complete intersections*, Explicit birational geometry of 3-folds, London Math. Soc. Lecture Note Ser., vol. 281, Cambridge Univ. Press, Cambridge, 2000, pp. 101–173. MR 1798982 (2001k:14089)
22. V. A. Iskovskih and Yuri I. Manin, *Three-dimensional quartics and counterexamples to the Lüroth problem*, Mat. Sb. (N.S.) **86(128)** (1971), 140–166. MR 0291172 (45 #266)
23. Vasily A. Iskovskikh, *Birational automorphisms of three-dimensional algebraic varieties*, Current problems in mathematics, Vol. 12 (Russian), VINITI, Moscow, 1979, pp. 159–236, 239 (loose errata). MR 537686 (81i:14026c)
24. Vasily A. Iskovskikh and Yu. G. Prokhorov, *Fano varieties*, Algebraic geometry, V, Encyclopaedia Math. Sci., vol. 47, Springer, Berlin, 1999, pp. 1–247. MR 1668579 (2000b:14051b)
25. Vasily A. Iskovskikh and Alexander V. Pukhlikov, *Birational automorphisms of multidimensional algebraic manifolds*, J. Math. Sci. **82** (1996), no. 4, 3528–3613, Algebraic geometry, 1. MR 1426395 (98e:14009)
26. Anne-Sophie Kaloghiros, *Relations in the Sarkisov program*, arxiv:1203.3767 (2012), To appear in Compositio Mathematica.
27. Yujiro Kawamata, *Divisorial contractions to* 3*-dimensional terminal quotient singularities*, Higher-dimensional complex varieties (Trento, 1994), de Gruyter, Berlin, 1996, pp. 241–246. MR 1463182 (98g:14005)
28. János Kollár and Shigefumi Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, Cambridge, 1998, With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original. MR 1658959 (2000b:14018)

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